

# A Diophantine Frobenius problem related to Riemann surfaces

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## Abstract

We obtain sharp upper and lower bounds on a certain four-dimensional Frobenius number determined by a prime pair  $(p, q)$ ,  $2 < p < q$ , including exact formulae for two infinite subclasses of such pairs. Our work is motivated by the study of compact Riemann surfaces which can be realized as a semi-regular  $pq$ -fold coverings of surfaces of lower genus. In this context, the Frobenius number is (up to an additive translation) the largest genus in which no surface is such a covering. In many cases it is also the largest genus in which no surface admits an automorphism of order  $pq$ . The general  $t$ -dimensional Frobenius problem ( $t \geq 3$ ) is *NP*-hard, and it may be that our restricted problem retains this property.<sup>1</sup>

## 1 Introduction

A set of integers  $\{a_1, a_2, \dots, a_t\}$ ,  $t \geq 2$ , with  $a_i > 1$  and  $\gcd = 1$ , has a *Frobenius number*

$$g(\{a_1, a_2, \dots, a_t\}),$$

which is the largest positive integer not representable in the form  $k_1 a_1 + k_2 a_2 + \dots + k_t a_t$ , where each  $k_i$  is a nonnegative integer. It is a simple exercise to show that  $g(\{a_1, a_2, \dots, a_t\})$  exists under the stated conditions. Finding  $g(\{a_1, \dots, a_t\})$  for a given set  $\{a_1, \dots, a_t\}$  is the linear Diophantine problem of Frobenius [11]. In 1884, J.J. Sylvester established the formula

$$g(\{a_1, a_2\}) = a_1 a_2 - a_1 - a_2 \tag{1.1}$$

for the *two-dimensional* Frobenius number [12]. In 1990, it was shown by F. Curtis [2] that, for  $t \geq 3$ , there is no finite set of polynomials  $\{f_1, \dots, f_k\}$  in  $t$  variables such that, for each  $t$ -tuple  $\{a_1, a_2, \dots, a_t\}$  with greatest common divisor 1,  $g(\{a_1, a_2, \dots, a_t\}) = f_i(a_1, a_2, \dots, a_t)$  for some  $i$ . Algorithms for computing  $t$ -dimensional Frobenius numbers exist [11], but the problem (for variable  $t \geq 3$ ) is *NP*-hard [10].

Throughout the paper,  $p, q$  will be primes satisfying  $2 < p < q$  with  $p', q'$  denoting the integers  $(p-1)/2$  and  $(q-1)/2$ , respectively. The four integers

$$d_0 = pq, \quad d_1 = p'q, \quad d_2 = pq', \quad d_3 = (pq-1)/2, \tag{1.2}$$

have  $\gcd = 1$ , so they determine a four-dimensional Frobenius number

$$g_{pq} = g(\{d_0, d_1, d_2, d_3\}). \tag{1.3}$$

The significance of the number  $g_{pq}$  in (1.3) is that

$$g_{pq} - pq + 1$$

is the largest integer such that no compact Riemann surface of that genus is a semi-regular  $pq$ -fold cover of some other surface. This is explained in Section 3. A closely related quantity of interest to us is  $\nu_{pq}$ , the largest integer such that no compact Riemann surface of that genus has an automorphism group that is cyclic of order  $pq$ .  $\nu_{pq}$  is called the *largest non-genus* of the group  $\mathbb{Z}_{pq}$ . As a special case of Theorem 3.3 we have

$$g_{pq} - pq + 1 \leq \nu_{pq} \leq g_{pq}. \tag{1.4}$$

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Our main results, listed in the next section, yield bounds for  $g_{pq}$ . When  $q$  is sufficiently large with respect to  $p$  we obtain exact formulas for  $g_{pq}$  as well as  $\nu_{pq}$ . At the other extreme we also give exact formulas for  $g_{pq}$ ,  $\nu_{pq}$  when  $q = p + 2$ .

More generally, as we describe in Section 3, there is a Frobenius number  $g_n$  so that  $g_n - n + 1$  is the largest possible genus for a compact Riemann surface that is not a semi-regular  $n$ -fold cover of another surface. For square-free odd  $n$  with  $s > 2$  prime factors, this will correspond to a more difficult  $2^s$ -dimensional Frobenius problem.  $\nu_n$ , the largest non-genus for the cyclic group  $\mathbb{Z}_n$ , has been found in the case of  $n = p^e$  for  $p$  prime by Kulkarni and Maclachlan in [7]. Kulkarni in [6] showed that, for an arbitrary finite group  $G$ , the genera where it is possible for a surface to admit  $G$  as an automorphism group form an arithmetic progression. He showed that there also exists a largest non-genus in this progression. These genera are studied with generating functions in [8].

## 2 The main results

Define the function

$$f_{p,q}(x, y, z, w) = xd_0 + yd_1 + zd_2 + wd_3, \quad (2.1)$$

where the integers  $d_i$  are defined at (1.2). A positive integer  $n$  is *representable* if  $n = f_{p,q}(x, y, z, w)$  for  $x, y, z, w$  nonnegative. The Frobenius number (1.3) is the largest non-representable integer. Since  $p$  and  $q$  are fixed in all our arguments, we henceforth put  $f = f_{p,q}$ , suppressing the subscripts and write  $g$  for  $g_{pq}$ .

We define integers  $\kappa, \kappa', \lambda, \lambda'$  as follows:

$$q = \kappa p + \lambda, \quad 1 \leq \lambda \leq p - 1 \quad (2.2)$$

$$q' = \kappa' p' + \lambda', \quad 0 \leq \lambda' \leq p' - 1. \quad (2.3)$$

The integers

$$G_0 \equiv f(p' - 1, p - 1, \kappa, -1), \quad G_1 \equiv G_0 - \lambda d_3, \quad G_2 \equiv G_0 - (p - 3)d_3 \quad (2.4)$$

play an important role.

**Theorem 2.1.** *The Frobenius number  $g$  satisfies*

$$(i) \quad G_2 \leq g \leq G_0$$

$$(ii) \quad g = G_0 \text{ if and only if } \kappa + \lambda \geq p$$

$$(iii) \quad g = G_2 \text{ if } p = 3 \text{ or if } (p, q) \text{ is a twin prime pair.}$$

We note that if  $p = 3$  then  $G_2 = G_0$  and  $\kappa + \lambda \geq 3$ , so that Theorem 2.1 parts (i) and (ii) each imply that  $g = G_0$  (and hence also  $g = G_2$ , as in part (iii)). For  $p > 3$ , the integer  $q = (p - 3)p + 1$ , if prime, is the largest such that  $\kappa + \lambda < p$ . Hence we obtain an easy corollary.

**Corollary 2.2.** *If  $q \geq (p - 3)p + 3$ ,  $g = G_0$ .*

When  $\kappa + \lambda < p$ , by Theorem 2.1,  $G_2 \leq g < G_0$ . These bounds can be tightened in some cases. To treat these cases, we introduce some more notation.

Note that  $\kappa$  and  $\lambda$  have opposite parity (otherwise  $q$  is not prime), and that  $\kappa' \geq \kappa$ . If  $\kappa + \lambda < p$ , then, in fact,  $\kappa + \lambda \leq p - 2$  and hence  $\lambda \leq p - 3$ . It follows that there is a unique nonnegative integer  $\tau < \lambda$  such that

$$\frac{\tau + 2}{\tau + 1} < \frac{p}{\lambda} < \frac{\tau + 1}{\tau}. \quad (2.5)$$

We allow  $\tau = 0$  so as to include the cases in which  $2 < \frac{p}{\lambda}$ . (It is also easy to see that  $\tau = \lfloor \lambda / (p - \lambda) \rfloor$ .) Every pair  $(p, q)$  with  $\kappa + \lambda < p$  belongs to one of two types:

$$\text{Type I: } \frac{\tau + 2}{\tau + 1} < \frac{p'}{\lambda'} \quad \text{Type II: } \frac{p'}{\lambda'} \leq \frac{\tau + 2}{\tau + 1}. \quad (2.6)$$

**Theorem 2.3.** *For a Type II pair, the Frobenius number  $g$  satisfies*

$$(i) \quad G_1 \leq g < G_0$$

(ii)  $g = G_1$  if and only if  $\kappa + \lambda \leq p - \lambda$ .

**Theorem 2.4.** For a Type I pair with  $\kappa + \lambda \leq p - \lambda$ , the Frobenius number  $g$  satisfies

(i)  $G_2 \leq g < G_1$

(ii)  $g = G_2$  if  $(p, q)$  is a twin prime pair.

The above theorems show where  $g$  lies in relation to  $G_0$ ,  $G_1$  and  $G_2$ . Figure 1 shows how these results are distributed over small prime pairs. The four displayed cases correspond to  $g = G_0$ ,  $g = G_1$ ,  $G_1 < g < G_0$  and  $G_2 \leq g < G_1$ , respectively.

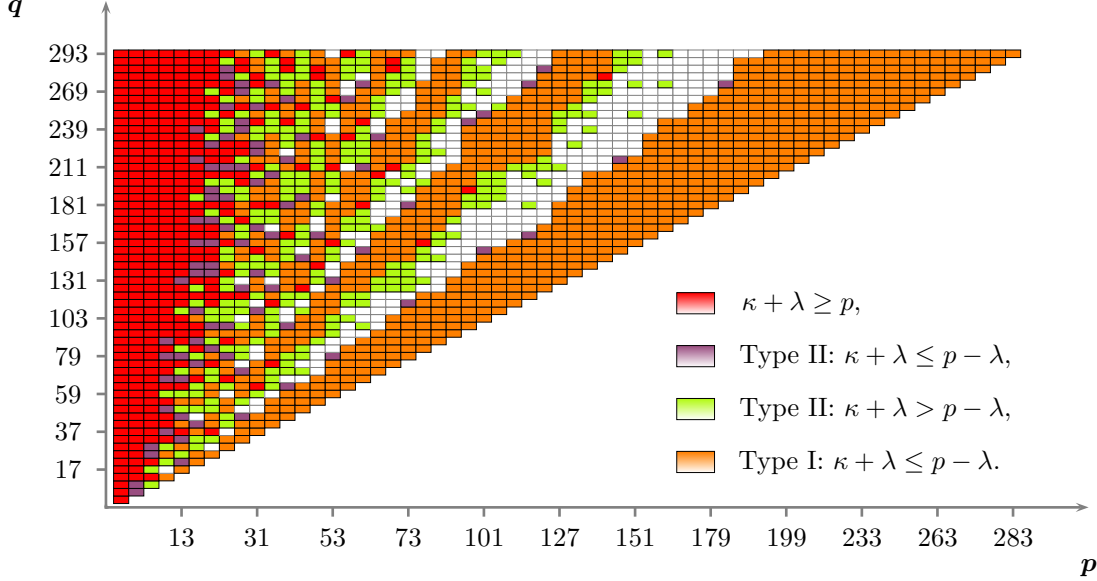


Figure 1:  $2 < p < q < 300$

With (1.4), we may translate the bounds on  $g$  into bounds on  $\nu_{pq}$ . We can do better in the case when  $\kappa + \lambda \geq p$ , where, by Theorem 2.1, we have  $g = G_0$ .

**Theorem 2.5.** For primes  $3 < p < q$ , with  $\kappa + \lambda \geq p$  and  $q \neq 2p - 1, 3p - 2$ , we have

$$\nu_{pq} = G_0 - pq + 1.$$

Thus,  $\nu_{pq}$  attains the lower bound of (1.4) in this case. For a twin prime pair,  $\nu_{pq}$  lies about halfway between the bounds of (1.4) (see Theorem 9.2). It appears that the upper bound is not attained for any prime pair.

The Type I pairs not covered by Theorem 2.4 (white in the Figure 1) are those for which  $p > \kappa + \lambda > p - \lambda$  (see Remark 1, Section 7). We plan to treat these pairs in a future paper. For now, we note that the formula for the Frobenius number  $g_{pq}$  depends on the number theoretic relationship between  $q/p$  and  $q'/p'$ . Making this dependence precise involves the continued fraction

$$\frac{q}{p} = q_1 + \frac{1}{q_2 + \frac{1}{q_3 + \frac{1}{\ddots + \frac{1}{q_{n-1} + \frac{1}{q_n}}}}}. \quad (2.7)$$

The condition  $\kappa + \lambda \geq p$ , appearing in Theorem 2.1(ii), is equivalent to  $q_1 + 1 \leq q'/p'$ . It appears that the next case is

$$q_1 + \frac{1}{q_2} \leq \frac{q'}{p'} < q_1 + 1$$

and that an exact, though more complicated, formula for  $g$  is also possible in this case. It seems likely that  $g_{pq}$  depends on where  $q'/p'$  lies in relation to the convergents of (2.7).

### 3 The motivating problems

If a compact Riemann surface  $X$  admits a finite group  $G$  of conformal *automorphisms*, the quotient space  $Y = X/G$  is itself a compact Riemann surface, and the quotient map  $\Phi : X \rightarrow Y$  is a holomorphic *branched covering map* of degree  $n = |G|$  (the order of  $G$ ). This means that  $\Phi$  is generically  $n$ -to-1 (or  $n$ -fold), but there is a finite subset  $B \subset Y$ , called the branch set, over which the fibers have cardinality strictly less than  $n$ . The *Riemann-Hurwitz relation*, a linear Diophantine equation, relates the topological data associated with  $\Phi$ , namely, the genera of the surfaces, the degree of the covering, and the cardinalities of the fibers over the branch set. It is a generalization of the multiplicative relation between the Euler characteristics of the surfaces,  $\chi(X) = n \cdot \chi(Y)$  which holds for  $n$ -fold *unbranched* covering maps. (See [3], Sections I.1 and I.2, for a fuller treatment of these ideas.)

Branched covering maps need not arise as quotient maps of group actions. Those that do must satisfy an extra regularity condition: for every  $y \in B$ , there exists a divisor  $n_y > 1$  of  $n$  such that the fiber over  $y$  consists of precisely  $n/n_y$  points, at which the  $n$  sheets of the covering come together in sets of  $n_y$ . The integers  $n_y$ ,  $y \in B$ , are called the *branching indices*, and the covering is called *semi-regular*. When  $\Phi : X \rightarrow Y$  is a semi-regular branched covering, the Riemann-Hurwitz relation is

$$2(\gamma - 1) = 2n(\eta - 1) + n \sum_{y \in B} \left(1 - \frac{1}{n_y}\right), \quad (3.1)$$

where  $\gamma, \eta$ , are the genera of  $X, Y$ , respectively,  $n$  is the degree of the covering, and  $n_y$ ,  $y \in B$  are the branching indices. If  $\Phi$  can be realized as the quotient map of a group action, the covering is called *regular*.

We now specialize to the case where  $n$ , the degree of the covering, is a square-free odd integer with  $s \geq 1$  distinct prime factors  $p_i$ ,  $i = 1, 2, \dots, s$ . The  $2^s$  divisors of  $n$  are in one-to-one correspondence with the set  $\mathcal{B}$  of binary bit strings of length  $s$ . Let  $I$  denote a bit string of length  $s$ , and  $\bar{0}, \bar{1}$  the bit strings consisting of all 0's, and all 1's, respectively. Let  $n_I$  denote the divisor of  $n$  associated with the bit string  $I$ , so that, for example,  $n_{\bar{0}} = 1$  and  $n_{\bar{1}} = n$ , and, more generally,  $p_i$  is a factor of  $n_I$  if and only if the  $i$ th bit of  $I$  is 1. Then (3.1) implies the Riemann-Hurwitz formula for an  $n$ -fold semi-regular branched covering is

$$\gamma + n - 1 = x_{\bar{0}}n + \sum_{I \neq \bar{0}} x_I \frac{n(n_I - 1)}{2n_I}, \quad (3.2)$$

where  $x_I$  ( $I \neq \bar{0}$ ) is the number of points in the branch set with branching index  $n_I$ , and  $x_{\bar{0}} = \eta$ , the genus of  $Y$ . The integers

$$d_{\bar{0}} = n, \quad d_I = \frac{n(n_I - 1)}{2n_I}, \quad I \in \mathcal{B}, I \neq \bar{0}$$

have  $\gcd = 1$ , so there is a  $2^s$ -dimensional Frobenius number  $g(\{d_I : I \in \mathcal{B}\})$ . By the general theory of branched coverings, there is surface  $X$  of genus  $\gamma$  which is an  $n$ -fold semi-regular covering if and only if there is a  $2^s$ -tuple  $(x_I)_{I \in \mathcal{B}}$  of nonnegative integers satisfying (3.2). It follows that there is a largest *non-genus* of a semi-regular  $n$ -fold covering, namely, the additive translate  $-n + 1 + g(\{d_I : I \in \mathcal{B}\})$  of the  $2^s$ -dimensional Frobenius number  $g(\{d_I : I \in \mathcal{B}\})$ .

**Problem I:** For every square-free odd  $n$  with  $s$  distinct prime factors, determine the largest non-genus of a semi-regular  $n$ -fold covering. This genus is  $g_n - n + 1$  where  $g_n$  is the  $2^s$ -dimensional Frobenius number  $g(\{d_I : I \in \mathcal{B}\})$ .

The case  $s = 1$  of Problem I follows immediately from the previous paragraph and Sylvester's formula (1.1) for the 2-dimensional Frobenius number.

**Proposition 3.1.** *Let  $p$  be an odd prime. The largest non-genus of a  $p$ -fold semi-regular branched covering is  $p'(p - 3) - p$ , the additive translate  $g(\{p, p'\}) - p + 1$  of the two dimensional Frobenius number  $g(\{p, p'\})$ .*

Note that this integer is  $< 0$  for  $p = 3, 5$ , so that there is a semi-regular 3- or 5- fold branched covering of every genus.

#### 3.1 Group actions

We now give a set of necessary and sufficient conditions for the existence of a *regular cyclic*  $n$ -fold branched covering  $\Phi : X \rightarrow Y$ , that is, a covering realizable as the quotient map of a cyclic group  $\mathbb{Z}_n$  of automorphisms

acting on the compact Riemann surface  $X$ . The conditions are a special case of a more general set of conditions for the existence of an action by an arbitrary finite group  $G$  of order  $n$ . The necessary and sufficient condition in the general case is the existence of a *partial monodromy presentation* of  $G$ , having a form dictated by the genus of  $Y$  and the branching indices. If the genus of  $Y$  is  $\eta$ , and the branching indices are  $r_1, r_2, \dots, r_k$ , the monodromy presentation of  $G$  must have  $2\eta + k$  generators  $a_1, b_1, \dots, a_\eta, b_\eta, c_1, \dots, c_k$ , where  $c_i$  has order  $r_i$  and, among other possible relations,

$$\prod_{i=1}^{\eta} [a_i, b_i] \prod_{j=1}^k c_j = 1, \quad (3.3)$$

where  $[a_i, b_i]$  denotes the commutator and 1 denotes the identity element in  $G$ . (For a fuller explanation of the general case, see, e.g., [1], or [9, Chapter III, Section 3], or [4, Section 1.7].)

**Lemma 3.2.** *Let  $n = p_1 \dots p_s$ ,  $s \geq 1$ , a square-free odd integer with prime factors  $p_1, \dots, p_s$ . Let  $\gamma \geq 0$ . Let  $(x_I \geq 0)_{I \in \mathcal{B}}$  be a  $2^s$ -tuple satisfying (3.2) for  $n, \gamma$ . There is a compact Riemann surface of genus  $\gamma$  admitting a group of automorphisms  $\mathbb{Z}_n$  such that quotient surface has genus  $x_{\bar{0}}$  and  $x_I$  points of branching indices  $n_I$ ,  $I \neq \bar{0}$ , if and only if the tuple  $(x_I \geq 0)_{I \in \mathcal{B}}$  satisfies the admissibility conditions:*

$$\sum_{I \in \mathcal{B}^i} x_I \neq 1, \quad i = 1, 2, \dots, s; \quad (3.4)$$

$$x_{\bar{0}} + \sum_{I \in \mathcal{B}^i} x_I \neq 0, \quad i = 1, 2, \dots, s, \quad (3.5)$$

where  $\mathcal{B}^i \subset \mathcal{B}$  is the set of bit strings of length  $s$  whose  $i$ th bit is 1.

*Proof.* A partial monodromy presentation of  $\mathbb{Z}_n$  dictated by the tuple  $(x_I \geq 0)_{I \in \mathcal{B}}$  would have  $2x_{\bar{0}}$  generators  $a_0, b_0, \dots, a_{x_{\bar{0}}}, b_{x_{\bar{0}}}$  of unspecified order, and  $x_I$  generators  $c_I$  of order  $n_I$  for each  $I \in \mathcal{B}$ ,  $I \neq \bar{0}$ . Since all commutators are trivial in an abelian group, the elements  $a_j, b_j$  can be omitted from the relation (3.3). If the  $i$ th condition in (3.4) fails, the group product on the left-hand side of (3.3) would contain exactly one element of order divisible by  $p_i$ , and hence could not be equal to the identity. If the  $i$ th condition in (3.5) fails, the generating set would contain no elements of order divisible by  $p_i$ , a contradiction. This proves the necessity of the conditions. To prove sufficiency of the conditions, one verifies that a partial monodromy presentation of  $\mathbb{Z}_n$  can be constructed in all other cases; this is left as an exercise.  $\square$

If there exist tuples  $(x_I \geq 0)_{I \in \mathcal{B}}$  satisfying (3.2) for some  $\gamma, n$ , but none of them satisfy all the admissibility conditions in (3.4) and (3.5), then  $\gamma$  is the genus of an  $n$ -fold semi-regular covering, but a *non-genus* for a  $\mathbb{Z}_n$  action. There exists a largest non-genus of a  $\mathbb{Z}_n$  action [6], and it must be at least as large as the largest non-genus of an  $n$ -fold semi-regular covering.

**Problem II:** For every square-free odd  $n$ , determine the largest non-genus  $\nu_n$  of  $\mathbb{Z}_n$ .

**Theorem 3.3.** *Let  $n$  be a square-free odd integer with  $s \geq 1$  distinct prime factors. With  $\nu_n$  denoting the largest non-genus of a  $\mathbb{Z}_n$  action, and  $g_n$  the Frobenius number  $g(\{d_I : I \in \mathcal{B}\})$ , we have*

$$g_n - n + 1 \leq \nu_n \leq g_n. \quad (3.6)$$

*Proof.* The left-hand inequality is clear:  $\gamma = g_n - n + 1$  is the largest integer such no  $2^s$ -tuple  $(x_I)_{I \in \mathcal{B}}$  (admissible or not) satisfies (3.2). Hence  $\nu_n$  must be at least as large as  $g_n - n + 1$ . For the right-hand inequality, let  $(x_I)_{I \in \mathcal{B}}$  be a nonnegative  $2^s$ -tuple satisfying (3.2) for some  $\gamma$ . The tuple obtained from  $(x_I)$  by replacing the final coordinate  $x_{\bar{1}}$  with  $x_{\bar{1}} + 2$  satisfies (3.2) with  $\gamma$  replaced by  $\gamma + n - 1$ . Moreover, the new tuple satisfies the admissibility conditions (3.4) and (3.5). Thus, if there is a surface of genus  $\gamma$  which is an  $n$ -fold semi-regular covering, there is a surface of genus  $\gamma + n - 1$  which admits a  $\mathbb{Z}_n$  action. Consequently,  $\nu_n$  is no larger than  $g_n - n + 1 + n - 1 = g_n$ .  $\square$

In the case  $s = 1$ , the Riemann-Hurwitz relation is

$$\gamma + p - 1 = x_0 p + x_1 p' \quad (3.7)$$

and the admissibility conditions are simply  $x_0 + x_1 \neq 0$  and  $x_1 \neq 1$ . It is easy to verify that there is just one solution of (3.7) when  $\gamma = g(\{p, p'\})$ , namely, the inadmissible pair  $(x_0, x_1) = (p' - 1, 1)$ . Thus the largest

non-genus of a  $\mathbb{Z}_p$  action is strictly greater than the largest non-genus of a semi-regular  $p$ -fold covering (cf. Proposition 3.1). In fact it is known ([7]) that  $\nu_p = g(\{p, p'\}) = g_p$ . This shows that the upper bound in (3.6) can be attained. We conjecture that  $s = 1$  is the only case in which this occurs. We shall show in Section 9 that when  $s = 2$ , the lower bound in (3.6) is attained for infinitely many  $n = pq$ .

A group of square-free order is either cyclic or metacyclic (see, e.g., [5], Theorem 9.4.3). A metacyclic group has a normal cyclic subgroup with a cyclic factor group. If  $s = 1$ , the only possible group is  $\mathbb{Z}_p$ . If  $s = 2$ , there is a (nonabelian) metacyclic group (of order  $pq$ ) if and only if  $p$  is a divisor of  $q - 1$ . Such a group contains no elements of order  $pq$ , hence the quotient map has no branching indices equal to  $pq$ , and the corresponding Frobenius problem is 3-, not 4-dimensional. The admissibility conditions for a partial monodromy presentation are (naturally) different. A formula for the largest non-genus of a metacyclic group action of order  $pq$  is given by the second author in [13]. In section 9 we give a formula for the largest non-genus of  $\mathbb{Z}_{pq}$  which, given  $p$ , is valid for all but finitely many  $q > p$ .

Henceforth we treat Problems I and II exclusively for  $n$  a product of two distinct primes. Until the last section, we revert to the purely number theoretic question of determining the 4-dimensional Frobenius number  $g = g(\{d_0, d_1, d_2, d_3\})$ , with the  $d_i$  as defined at (1.2).

## 4 Representability of integers $> G_0$

To prove that a certain integer  $m$  is the Frobenius number  $g$ , we need to establish that (a)  $m$  is not representable as  $f(x, y, z, w)$  for any quadruple  $(x, y, z, w)$  of nonnegative integers; and (b) all integers  $> m$  are representable in this way. For (b), it suffices to show that all integers in the closed interval  $[m + 1, m + d_1]$  are representable, since if  $f(x, y, z, w)$  is a nonnegative representation of  $k \in [m + 1, m + d_1]$ , then  $f(x, y + l, z, w)$  is a nonnegative representation of  $k + ld_1$ , for any  $l \geq 0$ . In this and subsequent sections we apply this method to  $m = G_0, G_1$  and  $G_2$ , as they are defined at (2.4). Having applied the method to  $G_0$ , it will be possible to reuse much of the work in the treatment of  $G_1$  and  $G_2$ .

For  $x, y, z, w \in \mathbb{Q}$ , the equation  $f(x, y, z, w) = 0$  determines a three dimensional vector subspace of  $\mathbb{Q}^4$ , whose span is the hyperplane orthogonal to the vector  $(d_0, d_1, d_2, d_3)$ . It has an obvious basis consisting of the three vectors

$$(d_1, -d_0, 0, 0), \quad (0, d_2, -d_1, 0), \quad (0, 0, d_3, -d_2).$$

It is easy to show that

$$e_0 = (p', -p, 0, 0) \tag{4.1}$$

$$e_1 = (p', 0, 1, -p) \tag{4.2}$$

$$e_2 = (q', 1, 0, -q), \tag{4.3}$$

is also a basis. This basis is convenient since (an exercise shows) if there are integer quadruples  $(x, y, z, w)$  and  $(x', y', z', w')$  such that  $f(x, y, z, w) = f(x', y', z', w')$ , then the vector  $(x - x', y - y', z - z', w - w')$  is an integer linear combination of  $e_0, e_1$  and  $e_2$ . Thus, since  $f$  is linear,  $f(x, y, z, w) = f(x', y', z', w')$  if and only if  $(x', y', z', w') = (x, y, z, w) + \alpha e_1 + \beta e_2 + \gamma e_3$ , for some  $\alpha, \beta, \gamma \in \mathbb{Z}$ .

**Proposition 4.1.** *All integers  $> G_0$  are representable.*

To prove this, we show that for each integer  $n$  in the closed interval  $[G_0 + 1, G_0 + d_1]$ , a nonnegative quadruple  $(x, y, z, w)$  exists such that  $f(x, y, z, w) = n$ . We first construct quadruples (possibly with negative entries) representing the integers in  $[G_0 + 1, G_0 + d_1]$  and then show that they can be altered, if necessary, by adding an integer linear combination of the vectors  $e_0, e_1, e_2$ , so that they become *nonnegative* quadruples. We will make use of the following easily verified facts:

$$f(0, -1, 0, 1) = q' \tag{4.4}$$

$$f(0, 0, -1, 1) = p' \tag{4.5}$$

$$f(1, 0, 0, -2) = 1. \tag{4.6}$$

We start by obtaining a nonnegative representation of  $G_0 + 1$ , using  $f(e_1) = 0$  and (4.6):

$$G_0 + 1 = f(p' - 1, p - 1, \kappa, -1) - f(p', 0, 1, -p) + f(1, 0, 0, -2) = f(0, p - 1, \kappa - 1, p - 3). \tag{4.7}$$

We proceed to show that  $G_0 + 1 + t$  has a nonnegative representation for all  $t \in [0, d_1 - 1]$ .

Let an integer  $t \in [0, d_1 - 1]$  be represented with the division algorithm as

$$t = aq' + bp' + c, \quad \text{with } a \geq 0 \text{ maximal, } b \geq 0, \quad 0 \leq c \leq p' - 1. \quad (4.8)$$

The triple  $(a, b, c)$  is uniquely determined by  $t$  and conversely.

**Lemma 4.2.** *If  $t \in [0, d_1 - 1]$  has the representation (4.8), then*

$$(i) \ a \leq p - 1;$$

$$(ii) \ a = p - 1 \implies b = 0;$$

$$(iii) \ b \leq \kappa';$$

$$(iv) \ b = \kappa' \implies c < \lambda' \implies p > 3;$$

$$(v) \ b \geq \kappa \implies a \leq p - 2.$$

*Proof.* (i) and (v): If  $a \geq p$ , or if  $a = p - 1$  and  $b \geq \kappa$ , then  $t \geq p'q = d_1$ , contrary to assumption. (ii): If  $a = p - 1$  and  $t \leq p'q$ , then  $bp' + c \leq p' - 1$ , which implies  $b = 0$ . (iii): If  $b > \kappa'$ ,  $a$  is not maximal. (iv): If  $b = \kappa'$  and  $c \geq \lambda'$ ,  $a$  is not maximal. When  $p = 3$ ,  $\lambda' = 0$  and hence  $c < \lambda'$  is impossible.  $\square$

It follows from (4.4)-(4.6) that, for  $t \in [0, d_1 - 1]$ ,

$$G_0 + 1 + t = f(0, p - 1, \kappa - 1, p - 3) + a \cdot f(0, -1, 0, 1) + b \cdot f(0, 0, -1, 1) + c \cdot f(1, 0, 0, -2).$$

Thus  $G_0 + 1 + t = f(x, y, z, w)$ , where

$$x = c \quad (4.9)$$

$$y = p - 1 - a \quad (4.10)$$

$$z = \kappa - 1 - b \quad (4.11)$$

$$w = p - 3 - 2c + a + b. \quad (4.12)$$

By definition,  $x \geq 0$ . By Lemma 4.2(i),  $y \geq 0$ .  $w \geq 0$  because  $c \leq p' - 1$  is equivalent to

$$p - 3 - 2c \geq 0. \quad (4.13)$$

Thus  $z$  is the only component of the quadruple  $(x, y, z, w)$  which might be negative (if  $b \geq \kappa$ ). If this is the case,

$$b = \kappa + s \quad \text{for some } 0 \leq s \leq \kappa' - \kappa. \quad (4.14)$$

The upper bound on  $s$  is a consequence of Lemma 4.2(iii). We now show that there is always an integer linear combination of the vectors (4.2) and (4.3), which, when added to the quadruple defined by (4.9) - (4.12), yields a nonnegative quadruple. The argument will be divided into three parts (Lemmas 4.3, 4.4 and 4.5), according to whether  $s$  is, respectively, less than, equal to, or greater than  $\kappa' - \kappa - 1$ .

For notational convenience, we define the quadruple

$$e(u, v) \equiv (u - 1)e_2 + (v + 1)e_1, \quad (4.15)$$

where  $e_1, e_2$  are the vectors (4.2) and (4.3), respectively, and  $u, v \in \mathbb{Z}$ .

**Lemma 4.3.** *If  $s < \kappa' - \kappa - 1$ ,*

$$(i) \ \kappa + s - (s + 1)p \geq 0;$$

$$(ii) \ (x', y', z', w') = (x, y, z, w) + e(1, s) \text{ is a nonnegative quadruple.}$$

*Proof.* From  $q' \geq \kappa' p'$  we obtain  $q - 1 \geq \kappa'(p - 1) \iff \kappa' - 1 \geq \kappa' p - q = (\kappa' - \kappa)p + \kappa p - q$ , and hence

$$\kappa' - 1 \geq (\kappa' - \kappa)p - \lambda.$$

It follows from this that

$$\kappa' - 1 - l \geq (\kappa' - \kappa)p - \lambda - l, \quad \text{for } l \geq 0.$$

In particular, since  $\lambda \leq p - 1$ ,

$$\kappa' - 1 - l \geq (\kappa' - \kappa)p - lp \quad \text{if } l \geq 1. \quad (4.16)$$

Putting  $l = \kappa' - \kappa - s - 1 \geq 1$  in (4.16), we obtain (i). To prove (ii), we have  $x' > x \geq 0$ ,  $y' = y \geq 0$ , and  $z' = -(s + 1) + (s + 1) = 0$ . We need only show that

$$w' = p - 3 - 2c + a + b - (s + 1)p \quad (4.17)$$

is nonnegative. Recalling that  $b = \kappa + s$ , and using (4.13), we obtain

$$w' \geq \kappa + s - (s + 1)p.$$

Thus  $w' \geq 0$  is a consequence of (i). □

If  $s = \kappa' - \kappa - 1$ , then  $w \geq (s + 1)p$  easily implies that the fourth coordinate of  $(x, y, z, w) + e(1, s)$  is positive. The following lemma treats the case  $w < (s + 1)p$ , where the fourth coordinate of  $(x, y, z, w) + e(1, s)$  is negative.

**Lemma 4.4.** *If  $s = \kappa' - \kappa - 1$  and  $w < (s + 1)p$ ,*

(i)  $c - \lambda' \geq 0$ ;

(ii)  $(x', y', z', w') = (x, y, z, w) + e(0, \kappa' - 1)$  is a nonnegative quadruple.

*Proof.*  $w = p - 3 - 2c + a + \kappa + s < (s + 1)p$  is equivalent to

$$\begin{aligned} 2c &> \kappa - 3 - s(p - 1) \\ c &> (\kappa - 3)/2 - sp' \\ c &\geq (\kappa - 1)/2 - sp'. \end{aligned}$$

Putting  $s = \kappa' - \kappa - 1$ , we have

$$\begin{aligned} c &\geq (\kappa - 1 + p - 1)/2 - (\kappa' - \kappa)p' \\ &\geq (\kappa + \lambda - 1)/2 - (\kappa' - \kappa)p' \\ &= \lambda', \end{aligned}$$

where we have used  $\lambda \leq p - 1$  and Lemma 7.1. Thus (i) is proved.  $x' = c - \lambda'$  which is  $\geq 0$  by (i).  $y' = p - 1 - a - 1 \geq 0$  by Lemma 4.2(v).  $z' = -s - 1 + \kappa' = \kappa \geq 1$ . Finally,  $w' = w + q - k'p = w - (\kappa' - \kappa)p + \lambda$ . Since  $w \geq b = \kappa + s = \kappa' - 1$ , and  $\kappa' - 1 \geq (\kappa' - \kappa)p - \lambda$  by (4),

$$w' = w - (\kappa' - \kappa)p + \lambda \geq (\kappa' - \kappa)p - \lambda - (\kappa' - \kappa)p + \lambda = 0.$$

Thus (ii) is proved. □

**Lemma 4.5.** *If  $s = \kappa' - \kappa$ ,  $(x', y', z', w') = (x, y, z, w) + e(0, \kappa')$  is a nonnegative quadruple.*

*Proof.* By Lemma 4.2(iv),  $c < \lambda'$ . Since both  $c$  and  $\lambda'$  are  $\leq p' - 1$ ,

$$1 \leq \lambda' - c \leq p' - 1. \quad (4.18)$$



We have

$$x' = c - q' + (\kappa' + 1)p' = p' - (\lambda' - c) \quad (4.19)$$

$$y' = p - 2 - a \quad (4.20)$$

$$z' = -(s + 1) + \kappa' + 1 = \kappa \quad (4.21)$$

$$\begin{aligned} w' &= p - 3 - 2c + a + \kappa' + q - (\kappa' + 1)p \\ &= -3 - 2c + a + q - \kappa'(p - 1) \\ &= -2 - 2c + a + q - 1 - \kappa'(p - 1) \\ &= -2 - 2c + a + 2(q' - \kappa'p') \\ &= -2 - 2c + a + 2\lambda' \\ &= 2(\lambda' - c) - 2 + a \end{aligned} \quad (4.22)$$

$x', w' \geq 0$  by (4.18).  $y' \geq 0$  by Lemma 4.2(v). Finally  $z' = \kappa \geq 1$ .  $\square$

Lemmas 4.3, 4.4 and 4.5 together constitute a proof of Proposition 4.1 which implies that  $g \leq G_0$  for all prime pairs  $2 < p < q$ .

## 5 Representability of $G_0$

In this section we prove

**Proposition 5.1.**  *$G_0$  is representable if and only if  $\kappa + \lambda < p$ .*

Suppose  $\kappa + \lambda < p$ , and put  $(x', y', z', w') = e(0, \kappa' - 1) + (p' - 1, p - 1, \kappa, -1)$ . Since

$$e(0, \kappa' - 1) = (-q' + \kappa'p', -1, \kappa', q - \kappa'p) = (-\lambda', -1, \kappa, \lambda) \quad (5.1)$$

(the last equality being a consequence of  $\kappa' = \kappa$ ), it easily verified that  $(x', y', z', w')$  is a nonnegative quadruple representing  $G_0$ .

To prove the necessity of the condition, we employ a number theoretic lemma whose proof is a simple exercise.

**Lemma 5.2.** *Let  $m, n$  be relatively prime integers. If  $am + bn = a'm + b'n$  for any  $a, a', b, b' \in \mathbb{Z}$  then there exists an integer  $l$  such that  $a' = a - ln$  and  $b' = b + lm$ .*

**Proposition 5.3.** *If  $(x_0, y_0, z_0, w_0)$  and  $(x, y, z, w)$  are integer quadruples such that  $f(x_0, y_0, z_0, w_0) = f(x, y, z, w)$ , there exists an integer  $l$  such that the system*

$$\begin{cases} x_0p + (y_0 + w_0)p' - lq' &= xp + (y + w)p' \\ z_0p + w_0 + lq &= zp + w. \end{cases}$$

*is satisfied.*

*Proof.* Using (2.1) and (1.2), we have

$$\begin{aligned} f(x_0, y_0, z_0, w_0) &= x_0pq + y_0p'q + z_0pq' + w_0(p'q + q') \\ &= q[x_0p + (y_0 + w_0)p'] + q'[z_0p + w_0]. \end{aligned} \quad (5.2)$$

Since  $q$  and  $q'$  are relatively prime, we can apply Lemma 5.2 with  $a, b$  being the two expressions in square brackets in (5.2).  $\square$

We now resume the proof of Proposition 5.1. Suppose that  $G_0 = f(x, y, z, w)$  with  $x, y, z, w$  nonnegative, and further suppose (for a contradiction) that  $\kappa + \lambda \geq p$  (equivalently,  $\kappa' - \kappa > 0$ ). Using Proposition 5.3 with  $(x_0, y_0, z_0, w_0) = (p' - 1, p - 1, \kappa, -1)$ , there exists an integer  $l$  such that

$$\begin{cases} (p' - 1)p + (p - 2)p' - lq' &= xp + (y + w)p' \\ \kappa p - 1 + lq &= zp + w. \end{cases} \quad (5.3)$$

The second equation implies  $l \geq 0$ , since the right hand side is nonnegative (by assumption) and  $\kappa p - 1 < q$ . To simplify the system (5.3), we express  $q'$  in terms of  $p'$  and  $p$ . We have

$$\begin{aligned} q' &= \kappa' p' + \lambda' \\ &= (\kappa' - 2\lambda')p' + 2\lambda'p' + \lambda' \\ &= (\kappa' - 2\lambda')p' + \lambda'p, \end{aligned}$$

where, at the last step, we use  $p = 2p' + 1$ . By Lemma 7.1,  $2\lambda' = \kappa + \lambda - 1 - (\kappa' - \kappa)(p - 1)$ , and hence

$$q' = Bp' + \lambda'p, \quad (5.4)$$

where

$$B = (\kappa' - \kappa)p - \lambda + 1. \quad (5.5)$$

Since  $\kappa' - \kappa > 0$  and  $\lambda < p$ ,  $B$  is positive.

Using (5.4), the first equation of (5.3) becomes

$$(p' - 1 - l\lambda')p + (p - 2 - lB)p' = xp + (y + w)p'. \quad (5.6)$$

Since  $p$  and  $p'$  are relatively prime, Lemma 5.2 applies to the left hand side, with  $a$  and  $b$  being the two expressions in parentheses. Hence there exists  $t \in \mathbb{Z}$  such that

$$\begin{aligned} x &= p' - 1 - l\lambda' + tp' \\ y + w &= p - 2 - lB - tp. \end{aligned} \quad (5.7)$$

The first equation implies  $t \geq 0$  (otherwise  $x < 0$ ) and the second that  $t \leq 0$  (otherwise  $y + w < 0$ ). Hence  $t = 0$ . Putting  $q = \kappa p + \lambda$  into the second equation of the system (5.3), we see that  $w \equiv l\lambda - 1 \pmod{p}$ . By (5.7),  $y + w \leq p - 2$  and in particular, since  $y$  and  $w$  are nonnegative,  $w \leq p - 2$ . The only possibility is  $w = l\lambda - 1$ . Hence

$$\begin{aligned} y &= p - 2 - lB - l\lambda + 1 \\ &= p - 1 - l[(\kappa' - \kappa)p + 1] \\ &\geq p - 1 - l(p + 1) \end{aligned} \quad (5.8)$$

(since  $\kappa' - \kappa > 0$ ).  $y \geq 0$  requires  $l = 0$ , and  $w \geq 0$  requires  $l > 0$ , a contradiction.

This completes the proof of Proposition 5.1. Combining this with Proposition 4.1, we obtain Theorem 2.1(ii).

## 6 Representability of integers $> G_1$

In this section and the next we recycle, as far as possible, the arguments in Sections 4 and 5, replacing  $G_0$  by  $G_1$ . Since  $G_1 = G_0 - \lambda d_3$ , we attempt this by simply reducing the fourth coordinate of each quadruple by  $\lambda$ . The obstruction, of course, is that some of the fourth coordinates thereby become negative.

**Proposition 6.1.** *If  $\kappa + \lambda \leq p - \lambda$ , all integers  $> G_1$  are representable.*

*Proof.* It suffices to show that there is a nonnegative quadruple representing each integer in the closed interval  $[G_1 + 1, G_1 + d_1]$ . We start with the representation  $G_1 + 1 = f(0, p - 1, \kappa - 1, p - 3 - \lambda)$ , obtained from (4.7) by subtracting  $\lambda$  from the fourth coordinate. The fact that  $\kappa + \lambda$  is odd and less than  $p$ , and that  $\kappa \geq 1$ , together imply that  $\lambda \leq p - 3$ ; thus this is a nonnegative representation. Representing  $t \in [0, d_1 - 1]$  by (4.8), we write  $G_1 + 1 + t = f(x, y, z, w)$ , where  $x, y, z$  are given by (4.9), (4.10), (4.11), respectively, and

$$w = p - 3 - 2c + a + b - \lambda, \quad (6.1)$$

which is obtained from (4.12) by subtracting  $\lambda$ .

If  $z \geq 0$ , the possible obstruction is  $w < 0$ . Then  $a \leq \lambda$  (since  $p - 3 - 2c \geq 0$ ). Adding (5.1) to  $(x, y, z, w)$  yields a nonnegative quadruple, provided  $c \geq \lambda'$ . If  $c < \lambda'$ ,

$$\begin{aligned} w &\geq p - 3 - 2(\lambda' - 1) + a + b - \lambda \\ &= p - 3 - (\kappa + \lambda - 3) + a + b - \lambda \\ &\geq p - \lambda - (\kappa + \lambda) \\ &\geq 0, \end{aligned}$$

contrary to the assumption that  $w < 0$ .

If  $z < 0$ ,  $b = \kappa$  and  $c < \lambda'$ . We write  $G_1 + 1 + t = f(x', y', z', w')$ , where  $x', y', z'$  are given by (4.19), (4.20), (4.21), respectively, and

$$w' = 2(\lambda' - c) - 2 + a - \lambda, \quad (6.2)$$

which is obtained from (4.22) by subtracting  $\lambda$ . The only possible obstruction is  $w' < 0$ . If this is the case, we add (5.1) to  $(x', y', z', w')$ , yielding the quadruple

$$\begin{aligned} x'' &= p' - 2\lambda' + c \\ y'' &= p - 3 - a \\ z'' &= 2\kappa \\ w'' &= 2(\lambda' - c) - 2 + a \end{aligned}$$

The assumption  $w' < 0$  implies  $a < \lambda \leq p - 3$ , so  $y'' \geq 0$ . Clearly  $z'' \geq 0$ .  $w'' \geq 0$  since it is equal to (4.22). If  $x'' < 0$  then  $c < 2\lambda' - p'$ . If this is the case,

$$\begin{aligned} w' &\geq 2(\lambda' - (2\lambda' - p' - 1)) - 2 + a - \lambda \\ &\geq 2p' - 2\lambda' - \lambda \\ &= 2p' - (\kappa + \lambda - 1) - \lambda \\ &= p - \lambda - (\kappa + \lambda). \\ &\geq 0, \end{aligned}$$

contradicting the assumption that  $w' < 0$ . Hence  $(x'', y'', z'', w'')$  is a nonnegative quadruple.  $\square$

**Corollary 6.2.** *If  $\kappa + \lambda \leq p - \lambda$  then  $g \leq G_1$ .*

To complete the proofs of Theorems 2.3 and 2.4, we need necessary and sufficient conditions for the representability of  $G_1$ , and conditions under which there is an integer  $> G_1$  which is not representable.

## 7 Representability of $G_1$ and $G_1 + \lambda'$

We need two preliminary results.

**Lemma 7.1.**  $\lambda' = \frac{\kappa + \lambda - 1}{2} - (\kappa' - \kappa)p'$ .

*Proof.* By definition  $q - \lambda = \kappa p$ , from which we obtain

$$\begin{aligned} (q - 1) - \lambda &= \kappa(p - 1) + \kappa - 1 \\ q' - \kappa p' &= \frac{\kappa + \lambda - 1}{2} \\ q' - \kappa' p' &= \frac{\kappa + \lambda - 1}{2} - (\kappa' - \kappa)p'. \end{aligned}$$

The left-hand side of the last equation is the definition of  $\lambda'$ .  $\square$

**Lemma 7.2.** *If  $\kappa + \lambda < p$ , then: (i)  $\kappa = \kappa'$ ; (ii)  $\lambda' = \frac{\kappa + \lambda - 1}{2} \geq 1$ ; (iii)  $\frac{p'}{\lambda'} < \frac{p}{\lambda}$ .*

*Proof.* Using the formula for  $\lambda'$  given in Lemma 7.1 and the assumption that  $\kappa + \lambda < p$ , we have  $\lambda' < p' - (\kappa' - \kappa)p'$ . Since  $\kappa' - \kappa \geq 0$  and  $\lambda' \geq 0$ , the only possibility is (i). The equality in (ii) follows from (i) and Lemma 7.1. The right-hand inequality follows from  $\kappa + \lambda \geq 3$ . To prove (iii), suppose that  $\frac{p'}{\lambda'} \geq \frac{p}{\lambda}$ . Then  $p'\lambda \geq p\lambda'$ , and hence, using (ii),

$$\begin{aligned} 2p'\lambda &\geq p(2\lambda') \\ (p - 1)\lambda &\geq p(\kappa + \lambda - 1) \\ -\lambda &\geq p(\kappa - 1) \geq 0, \end{aligned}$$

a contradiction.  $\square$

Suppose that  $G_1 = f(x, y, z, w)$  with  $x, y, z, w$  nonnegative. Using Proposition 5.3 with  $(x_0, y_0, z_0, w_0) = (p' - 1, p - 1, \kappa, -1 - \lambda)$ , there exists an integer  $l$  such that

$$\begin{cases} (p' - 1)p + (p - 2)p' - lq' &= xp + (y + w)p' \\ \kappa p - 1 - \lambda + lq &= zp + w. \end{cases} \quad (7.1)$$

The second equation implies  $l \geq 0$  (and the first that  $l$  cannot be too large). Imitating the argument leading from (5.3) to (5.7), we see that there exists an integer  $t \geq 0$  such that

$$x = p' - 1 - l\lambda' + tp' \quad (7.2)$$

$$y + w = p - 2 - l + \lambda(l - 1) - tp. \quad (7.3)$$

(We used  $B$  as defined at (5.5), but with  $\kappa' - \kappa = 0$ .) From the second equation of (7.1) (putting  $q = \kappa p + \lambda$ ), we see that  $w \equiv (l - 1)\lambda - 1 \pmod{p}$ . Then (7.3) yields  $y \equiv p - 1 - l \pmod{p}$ . Hence there exist  $\mu, \nu \in \mathbb{Z}$  such that

$$\begin{aligned} y &= \nu p + p - 1 - l \\ w &= \mu p + (l - 1)\lambda - 1. \end{aligned} \quad (7.4)$$

By (7.3),  $\mu + \nu = -t$ .  $\nu \geq 0$  from the assumption that  $y \geq 0$ . Provided that  $l \leq p - 1$  (we shall see shortly that this assumption is justified), we may add a suitable multiple of (4.1) to  $(x, y, z, w)$ , and so assume  $\nu = 0$ . Then  $\mu = -t$ . From (7.4) and the second equation of (7.1),

$$z = (l + 1)\kappa + t.$$

Thus a quadruple representing  $G_1$  has the general form

$$\begin{aligned} x &= p' - 1 - l\lambda' + tp' & (0 \leq t, \quad 0 \leq l \leq p - 1) \\ y &= p - (l + 1) \\ z &= (l + 1)\kappa + t \\ w &= -tp + (l - 1)\lambda - 1. \end{aligned} \quad (7.5)$$

( $t = l = 0$  yields the defining representation of  $G_1$ .)

**Proposition 7.3.**  $G_1$  is representable if and only if the pair is of Type I.

*Proof.* If the pair is of Type I, let  $t = \tau$  and  $l = \tau + 2$  in (7.5). Then  $x = (\tau + 1)p' - (\tau + 2)\lambda' - 1$  is nonnegative as a consequence of  $\frac{\tau+2}{\tau+1} < \frac{p'}{\lambda'}$ , and  $w = -\tau p + (\tau + 1)\lambda - 1$  is nonnegative as a consequence of  $\frac{p}{\lambda} < \frac{\tau+1}{\tau}$ . Obviously  $z \geq 0$ . It remains only to verify that  $l \leq p - 1$ , so that  $y \geq 0$ .  $\kappa + \lambda < p$  implies  $\lambda \leq p - 3$ , and  $\tau < \lambda$ , so  $l = \tau + 2 < \lambda + 2 \leq p - 1$ .

Suppose the pair is of Type II and  $t, l$  are nonnegative integers making (7.5) a nonnegative quadruple.  $x \geq 0, w \geq 0$  imply, respectively,

$$\frac{l}{t+1} < \frac{p'}{\lambda'}, \quad \text{and} \quad \frac{p}{\lambda} < \frac{l-1}{t}.$$

It follows that  $l > t + 1$ , and in particular,

$$\frac{t+2}{t+1} \leq \frac{l}{t+1} < \frac{p'}{\lambda'}, \quad \text{and} \quad \frac{p}{\lambda} < \frac{t+1}{t} < \frac{l-1}{t}. \quad (7.6)$$

Since the pair is of Type II, the left-hand inequality implies  $t > \tau$ , while the right-hand inequality, by the definition of  $\tau$ , implies that  $t \leq \tau$ , a contradiction.  $\square$

From Corollary 6.2 and Proposition 7.3, we obtain

**Corollary 7.4.** If the pair is of Type II with  $\kappa + \lambda \leq p - \lambda$ , then  $g = G_1$ . If the pair is of Type I with  $\kappa + \lambda \leq p - \lambda$ , then  $g < G_1$ .

The next proposition treats the remaining Type II pairs, and completes the proofs of all statements regarding  $G_0$  and  $G_1$  in Theorems 2.1, 2.3 and 2.4.

**Proposition 7.5.** *If the pair is of Type II with  $\kappa + \lambda > p - \lambda$ , then  $G_1 + \lambda'$  is not representable and hence  $g > G_1$ .*

*Proof.* Suppose  $G_1 + \lambda' = f(x, y, z, w)$  for a nonnegative quadruple  $(x, y, z, w)$ . A general form for  $(x, y, z, w)$  is produced from (7.5) by using (4.6) to write  $G_1 + \lambda' = G_1 + \lambda' \cdot f(1, 0, 0, -2)$ . Reducing the fourth coordinate of (7.5) by  $2\lambda' = \kappa + \lambda - 1$  (Lemma 7.2(ii)), and increasing the first by  $\lambda'$ , we obtain

$$\begin{aligned} x &= p' - 1 - (l - 1)\lambda' + tp' & (0 \leq t, \quad 0 \leq l \leq p - 1) \\ y &= p - (l + 1) \\ z &= (l + 1)\kappa + t \\ w &= -tp + (l - 2)\lambda - \kappa. \end{aligned} \tag{7.7}$$

The assumptions  $x \geq 0$  and  $w \geq 0$  imply almost the same inequalities as at (7.6), except that  $l$  is replaced  $l - 1$  where it occurs. Regardless, we arrive at the same contradiction ( $t > \tau$  and  $t \leq \tau$ ) which concluded the proof of Proposition 7.3.  $\square$

**Remark 1.** *For the remaining Type I pairs (having  $\kappa + \lambda > p - \lambda$  and colored white in Figure 1), both  $g < G_1$  and  $g > G_1$  are possible. A patient reader can verify, for example, that  $g < G_1$  for the pair (11, 17) and  $g > G_1$  for the pair (29, 103).*

## 8 The lower bound

It remains to prove that  $G_2$  is a universal lower bound on the Frobenius number, and that it is sharp if  $p = 3$  or if  $(p, q)$  is a twin prime pair.

**Proposition 8.1.**  *$G_2$  is not representable for any pair with  $\kappa + \lambda < p$ .*

*Proof.* Suppose  $(p, q)$  is a pair for which  $G_2$  is representable. Using Proposition 5.3 with  $(x_0, y_0, z_0, w_0) = (p' - 1, p - 1, \kappa, 2 - p)$ , there exists an integer  $l$  such that

$$\begin{cases} (p' - 1)p + p' - lq' &= xp + (y + w)p' \\ \kappa p + 2 - p + lq &= zp + w, \end{cases} \tag{8.1}$$

for nonnegative integers  $x, y, z, w$ . The second equation implies  $l \geq 0$ . Collecting the multiples of  $p$  and the multiples of  $p'$  on the left-hand side of the first equation, and using (5.4) and (5.5) with  $\kappa = \kappa'$ , and Lemma 5.2, we see that there exists  $t \in \mathbb{Z}$  such that

$$x = p' - 1 - l\lambda' + tp' \tag{8.2}$$

$$y + w = 1 + l(\lambda - 1) - tp. \tag{8.3}$$

(8.2) implies  $t \geq 0$ . Putting  $q = \kappa p + \lambda$  into the second equation of (8.1),

$$p(\kappa(l + 1) - 1) + l\lambda + 2 = zp + w. \tag{8.4}$$

It follows that  $w \equiv l\lambda + 2 \pmod{p}$ , and, using (8.3), that  $y \equiv -(l + 1) \pmod{p}$ . Hence there exist  $\mu, \nu \in \mathbb{Z}$  such that

$$\begin{aligned} y &= \nu p - (l + 1) & (\nu > 0) \\ w &= \mu p + l\lambda + 2. \end{aligned} \tag{8.5}$$

By (8.3),  $\mu = -t + \nu$ . From (8.4),  $z = \kappa(l + 1) - 1 + t + \nu$ .

Thus a quadruple representing  $G_2$  has the general form

$$\begin{aligned} x &= p' - 1 - l\lambda' + tp' & (t, l \geq 0) \\ y &= \nu p - (l + 1) & (\nu > 0) \\ z &= \kappa(l + 1) - 1 + t + \nu \\ w &= -(t + \nu)p + l\lambda + 2. \end{aligned} \tag{8.6}$$

( $t = l = 0$ ,  $\nu = 1$  yields the defining representation of  $G_2$ .) The requirements  $x \geq 0$  and  $w \geq 0$  imply

$$\frac{(t + \nu)p - 2}{\lambda} \leq l \leq \frac{(t + 1)p' - 1}{\lambda'}.$$

We show that this leads to a contradiction. Minimizing the left hand member of the inequality by taking  $\nu = 1$ , we obtain

$$\frac{(t + 1)p - 2}{\lambda} \leq \frac{(t + 1)p' - 1}{\lambda'}. \quad (8.7)$$

Since  $\kappa + \lambda < p$ ,  $\lambda' = (\lambda + (\kappa - 1))/2 \geq \lambda/2$ , and hence  $1/\lambda' \leq 2/\lambda$ , with equality if and only if  $\kappa = 1$ . Rearranging (8.7), we obtain

$$(t + 1) \left( \frac{p}{\lambda} - \frac{p'}{\lambda'} \right) \leq \left( \frac{2}{\lambda} - \frac{1}{\lambda'} \right),$$

which is a contradiction if  $\kappa = 1$ , since then the left-hand side is positive (Lemma 7.2 (iii)), while the right-hand side is 0. Hence assume  $\kappa > 1$ , and multiply both sides by  $\lambda\lambda' > 0$ . This yields

$$(t + 1)(p\lambda' - p'\lambda) \leq 2\lambda' - \lambda.$$

The right hand side is equal to  $\kappa - 1 > 0$ , and the left-hand side can be rewritten as

$$\frac{1}{2}(t + 1)(p(\kappa + \lambda - 1) - (p - 1)\lambda),$$

which simplifies to

$$\frac{1}{2}(t + 1)(p(\kappa - 1) + \lambda).$$

Thus we have

$$\frac{1}{2}(t + 1)(p(\kappa - 1)) < \kappa - 1.$$

Canceling the non-zero factor  $\kappa - 1$  leads to the contradiction

$$(t + 1)p < 2.$$

□

Thus  $G_2 \leq g$  for all pairs. The bound is attained if  $p = 3$ , by Theorem 2.1 (iii). The next proposition shows that the bound is also attained for twin prime pairs.

**Proposition 8.2.** *If  $(p, q)$  is a twin prime pair, all integers  $> G_2$  are representable.*

*Proof.* We adapt the proof of Proposition 4.1 (cf. Proposition 6.1). It suffices to show that the integers in the closed interval  $[G_2 + 1, G_2 + d_1]$  are representable. We start with the representation  $G_2 + 1 = \mathbf{f}(0, p - 1, 0, 0)$ , obtained from (4.7) by subtracting  $p - 3$  from the fourth coordinate, and using the fact that  $\kappa = 1$ . For twin pairs,  $q' = p' + 1$  and hence, from (4.4) and (4.5), we derive

$$\mathbf{f}(0, -1, 1, 0) = 1. \quad (8.8)$$

Let an integer  $t \in [0, d_1 - 1]$  be represented with the division algorithm as

$$t = aq' + b + c, \quad \text{with } 0 \leq a \leq p - 1 \text{ (} a \text{ maximal), } b, c \geq 0, \quad b + c \leq p'.$$

The bound on  $b + c$  comes from the maximality of  $a$  and the fact that  $q' = p' + 1$ .  $a$  and  $b + c$  are uniquely determined by  $t$  and conversely. It follows from (4.4), (4.6) and (8.8) that, for  $t \in [0, d_1 - 1]$ ,

$$G_2 + 1 + t = \mathbf{f}(0, p - 1, 0, 0) + a \cdot \mathbf{f}(0, -1, 0, 1) + b \cdot \mathbf{f}(0, -1, 1, 0) + c \cdot \mathbf{f}(1, 0, 0, -2).$$

Thus  $G_2 + 1 + t = \mathbf{f}(x, y, z, w)$ , where

$$\begin{aligned} x &= c \\ y &= p - 1 - (a + b) \\ z &= b \\ w &= a - 2c. \end{aligned}$$

If  $a \leq p'$ , then  $p - 1 - a \geq p'$  and we may assume  $c = 0$  by increasing  $b$ , if necessary, while maintaining  $y \geq 0$ . In fact,  $y \geq p - 1 - (p' + p') = 0$ ,  $w = a \geq 0$ , and we have a nonnegative quadruple. Hence suppose  $a = p' + s$ ,  $1 \leq s \leq p'$ . Let  $b = p' - i$  and  $c = p' - k$ ,  $0 \leq i, k \leq p'$ . Since  $b + c \leq p'$ ,  $i + k \geq p'$ . We claim there is a choice of  $i$  and  $k$  making  $(x, y, z, w)$  a non-negative quadruple. Clearly  $x, z \geq 0$  for all choices of  $i, k$ . If  $i \geq s$ ,  $w = s + 2k - p' \geq i + k + k - p' \geq p' + k - p' = k \geq 0$ . If  $i < s$ , put  $i' = i + (s - i) = s$  and  $k' = k - (s - i)$ , so that  $i' + k' = i + k \geq p'$ . Let  $b = p' - i'$  and  $c = p' - k'$ . Then

$$\begin{aligned} x &= p' - k' > p' - k \geq 0 \\ y &= i' - s = 0 \\ z &= p' - i' = p' - s \geq 0 \\ w &= i' + 2k' - p' \\ &= i' + k' + k' - p' \\ &\geq p' + k' - p' = k' \geq 0. \end{aligned}$$

□

**Remark 2.** We conjecture that  $g = G_2$  only if  $(p, q)$  is a twin prime pair.

This completes the proofs of Theorems 2.1, 2.3 and 2.4.

## 9 The largest non-genus of $\mathbb{Z}_{pq}$

We return to the motivating question of determining the largest non genus  $\nu_{pq}$  of a  $\mathbb{Z}_{pq}$  action (Problem II, Section 3.1,  $n = pq$ ). We show that given  $p > 3$ , the lower bound in (1.4) (and (3.6)) is attained for all but finitely many  $q > p$ . This is Theorem 2.5 which we restate here.

**Theorem 9.1.** For primes  $3 < p < q$ , with  $\kappa + \lambda \geq p$  and  $q \neq 2p - 1, 3p - 2$ , the largest non-genus of  $\mathbb{Z}_{pq}$  is

$$\nu_{pq} = G_0 - pq + 1,$$

where  $G_0$  is the integer defined at (2.4), equal to the Frobenius number  $g(\{d_0, d_1, d_2, d_3\})$ .

*Proof.* We re-visit the argument in Section 4, showing that the quadruples constructed there satisfy the admissibility conditions required by Lemma 3.2, or can be altered (by adding integer linear combinations of the vectors  $e_0, e_1, e_2$ ), so as to satisfy them. Bringing the notation in Lemma 3.2 into accord with that introduced in Sections 1 and 2, we put  $p = p_1$ ,  $q = p_2$ , and use  $x, y, z, w$  and  $d_0, d_1, d_2, d_3$  instead of  $x_{00}, x_{10}, x_{01}, x_{11}$ , and  $d_{00}, d_{10}, d_{01}, d_{11}$ , respectively. In this notation, conditions (3.4) and (3.5) are

$$y + w \neq 1, \quad z + w \neq 1, \tag{9.1}$$

$$x + y + w \neq 0, \quad x + z + w \neq 0. \tag{9.2}$$

It is convenient to replace the condition  $y + w \neq 1$  with the stronger condition  $y + w > 1$ . A nonnegative quadruple satisfying (9.1), (9.2) and  $y + w > 1$  will be called *strongly admissible*. The extra condition is imposed so that if  $(x, y, z, w)$  is strongly admissible, then  $(x, y+1, z, w)$  is admissible. With this guarantee, it is sufficient to produce strongly admissible representations of the integers in the closed interval  $[G_0 + 1, G_0 + d_1]$ .

Suppose first that the quadruple  $(x, y, z, w)$  as defined by (4.9) - (4.12) is nonnegative, that is, assume  $z \geq 0$ . One easily verifies that  $x + y + w \geq p - 1 > 0$ ,  $x + z + w \geq \kappa > 0$ ,  $y + w \geq p - 1 > 1$ . It remains to consider the possibility that

$$z + w = \kappa - 1 - b + p - 3 - 2c + a + b = 1.$$

This occurs if and only if

- (i)  $\kappa = 1$  and  $(a, b, c) = (1, b, p' - 1)$ ; or
- (ii)  $\kappa = 2$  and  $(a, b, c) = (0, b, p' - 1)$ .

$b \leq \kappa - 1$  by (4.11) and the assumption  $z \geq 0$ . Thus in (i),  $b = 0$ . The triple  $(1, 0, p' - 1)$  corresponds to the inadmissible quadruple  $(p' - 1, p - 2, 0, 1)$ .  $\kappa = 1$  is equivalent to  $p + 2 \leq q \leq 2p - 1$  or  $q' \leq 2p'$ . We have excluded  $q = 2p - 1$ , so we may assume  $q' < 2p'$ . It is easily verified that  $(p' - 1, p - 2, 0, 1) + e(0, 0)$  is strongly admissible. In case (ii),  $b = 0$  or  $1$  and the two triples  $(0, 0, p' - 1)$  and  $(0, 1, p' - 1)$  correspond to the inadmissible quadruples

$$(p' - 1, p - 1, 1, 0) \quad \text{and} \quad (p' - 1, p - 1, 0, 1), \quad (9.3)$$

respectively.  $\kappa = 2$  is equivalent to  $2p + 1 \leq q \leq 3p - 2$  or  $p \leq q' \leq 3p'$ . Since we have excluded  $q = 3p - 2$ , we may assume  $q' < 3p'$ . Addition of  $e(0, 1)$  makes both quadruples in (9.3) strongly admissible.

Now assume that  $z < 0$  in the quadruple  $(x, y, z, w)$  defined at (4.9)-(4.12). We re-visit the proofs of Lemmas 4.3, 4.4 and 4.5.

If  $s < \kappa' - \kappa - 1$ , Lemma 4.3(ii) produces the nonnegative quadruple

$$\begin{aligned} x' &= c + (s + 1)p' \\ y' &= p - 1 - a \\ z' &= 0 \\ w' &= p - 3 - 2c + a + \kappa + s - (s + 1)p. \end{aligned}$$

If this is inadmissible,  $w' \leq 1$ . By (4.13) and Lemma 4.3(i),  $w'$  is the sum of three nonnegative quantities:  $(p - 3 - 2c)$ ,  $a$ , and  $\kappa + s - (s + 1)p$ . Since  $p - 3 - 2c$  is even,  $w' \leq 1$  implies  $p - 3 - 2c = 0$ , equivalently,  $c = p' - 1$ . The other two quantities are either both 0, or one is 0 and the other 1. This yields three possible quadruples:  $(x', p - 1, 0, 0)$ , which is strongly admissible, and

$$(x', p - 1 - a, 0, 1), \quad a = 0, 1, \quad (9.4)$$

where  $x' = p' - 1 + (s + 1)p'$ . We show that these two quadruples cannot arise under the assumed conditions. They are supposed to represent the integers

$$G_0 + aq' + (\kappa + s)p' + p' - 1, \quad a = 0, 1.$$

Equating these two integers with the corresponding values of  $f$  on the two quadruples in (9.4), we obtain, for  $a = 0, 1$ ,

$$f(p' - 1, p - 1, \kappa, -1) + aq' + (\kappa + s)p' + p' - 1 = f(p' - 1 + (s + 1)p', p - 1 - a, 0, 1).$$

By the linearity of  $f$  this is equivalent to

$$\begin{aligned} 0 &= f(-(s + 1)p', a, \kappa, -2) + aq' + (\kappa + s)p' + p' - 1 \\ &= -(s + 1)p'd_0 + a(d_1 + q') + \kappa(d_2 + p') - 2d_3 + (s + 1)p' - 1 \\ &= -(s + 1)p'(d_0 - 1) + a(d_1 + q') + \kappa(d_2 + p') - 2d_3 - 1. \end{aligned} \quad (9.5)$$

The identities

$$d_1 + q' = d_2 + p' = d_3 \quad \text{and} \quad 2d_3 + 1 = d_0 \quad (9.6)$$

follow easily from the definitions of the  $d_i$ 's at (1.2). Thus (9.5) is equivalent to

$$\begin{aligned} d_0 &= -(s + 1)p'(2d_3) + (a + \kappa)d_3 \\ &= (a + \kappa - 2p'(s + 1))d_3 \\ &= (a + \kappa - (p - 1)(s + 1))d_3 \\ &= (a + \kappa + s - (s + 1)p + 1)d_3. \end{aligned} \quad (9.7)$$

The expression in parentheses on the right is equal to  $w' + 1 = 2$ . Thus (9.7) is equivalent to  $d_0 = 2d_3$ , contradicting the last identity in (9.6).

If  $s = \kappa' - \kappa - 1$ , Lemma 4.4(ii) produces the nonnegative quadruple

$$\begin{aligned} x' &= c - \lambda' \\ y' &= p - 2 - a \\ z' &= \kappa \\ w' &= p - 3 - 2c + a + [\kappa' - 1 - (\kappa' - \kappa)p + \lambda] \end{aligned} \quad (9.8)$$



We claim that the quantity in brackets in (9.8) is nonnegative. This is a consequence of

$$\begin{aligned}
q' &\geq \kappa' p' \\
q - 1 &\geq \kappa'(p - 1) \\
\kappa' - 1 &\geq \kappa' p - q \\
&= (\kappa' - \kappa)p + \kappa p - q \\
&= (\kappa' - \kappa)p - \lambda.
\end{aligned} \tag{9.9}$$

It follows that (9.8) is the sum of the three nonnegative quantities  $(p - 3 - 2c)$ ,  $a$ , and  $\kappa' - 1 - (\kappa' - \kappa)p + \lambda$ . If the quadruple is inadmissible,  $w' \leq 1$  and hence the even number  $p - 3 - 2c = 0$ , or equivalently,  $c = p' - 1$ . The other two quantities are either both 0, or one of them is 1 and the other is 0. If  $\kappa' - 1 - (\kappa' - \kappa)p + \lambda = 1$ , then  $a = 0$ , and we have the strongly admissible quadruple  $(p' - 1 - \lambda', p - 2, \kappa, 1)$ . If  $\kappa' - 1 - (\kappa' - \kappa)p + \lambda = 0$ , then reversing the chain of inequalities ending at (9.9), with inequalities replaced by equalities,  $q' = \kappa' p'$ . In that case,  $\lambda' = 0$  and we have the quadruples

$$(p' - 1, p - 2 - a, \kappa, 0), \quad a = 0, 1, \tag{9.10}$$

which are strongly admissible if  $\kappa > 1$  ( $p > 3$  is required here). If  $\kappa = 1$ , the quadruples in (9.10) are inadmissible, but  $\kappa + \lambda \geq p$  implies  $\lambda = p - 1$  and  $q = 2p - 1$ , which is excluded.

If  $s = \kappa' - \kappa$ , Lemma 4.5 produces the nonnegative quadruple (4.19) -(4.22). If this is inadmissible,  $w' \leq 1$ . By (4.18),  $w'$  is the sum of two nonnegative quantities,  $2(\lambda' - c) - 2$  and  $a$ . Since the former is even, it must be 0. Equivalently,  $c = \lambda' - 1$ . Hence there are two quadruples corresponding to  $a = 0, 1$ :

$$(p' - 1, p - 2, \kappa, 0), \quad (p' - 1, p - 3, \kappa, 1). \tag{9.11}$$

The latter is strongly admissible ( $p > 3$  is required), as is the former if  $\kappa > 1$ . If  $\kappa = 1$ ,  $(p' - 1, p - 2, \kappa, 0)$  is inadmissible. Then  $q' \leq 2p'$ , and in fact,  $q' < 2p'$  since  $q \neq 2p - 1$ . In that case, addition of  $e(0, 0)$  yields a strongly admissible quadruple. This concludes the proof of Theorem 9.1.  $\square$

For a twin prime pair, it is not difficult to show that the integer  $f(0, p - 1, 1, 0)$  has no other nonnegative representation, and hence no admissible representation. A straightforward argument, similar to the proof of Proposition 8.2, shows that the next  $d_1$  integers all have strongly admissible representations. This yields the following theorem, whose proof is omitted.

**Theorem 9.2.** *For a twin prime pair  $(p, q)$ ,  $p > 3$ , the largest non-genus of  $\mathbb{Z}_{pq}$  is*

$$\nu_{pq} = f(0, p - 1, 1, 0) - pq + 1 = G_2 + 1 + d_2 - pq + 1.$$

Hence for twin prime pairs,  $\nu_{pq}$  is about midway between the bounds of (1.4).

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